Twelve-Tone Composition in a Bayesian Framework: A Case Study in Stravinsky's "Libera me"

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Abstract

The present paper examines patterns of compositional decisions with respect to the choice of pitch materials in twelve-tone music. Based on the premise that Stravinsky, perhaps unconsciously, observed some specific rules of pitch organization, the paper analyzes "Libera me" from his *Requiem Canticles* as a case study, examines the patterns of his choice of tetrachordal subsets of the twelve-tone rows, and tries to formulate general rules of pitch organization in his twelve-tone music. Bayesian decision theory, that of Richard Jeffrey in particular, is employed to formulate the constraints in accordance with which Stravinsky made compositional decisions and his preference ranking of the tetrachords he chose for "Libera me" is deduced. A brief analysis of "Libera me" demonstrates that the preference ranking is relevant to the pitch organization of the composition.

COMPOSITIONAL CONSTRAINTS AND PC-SET PREFERENCES

In the field of research in atonal and twelve-tone music, while music theorists have proposed many rules of what Mead (1989) calls "taxonomies" such as those of the classification of pc-sets and transformations and some of them are widely used, we know still little about those rules that govern decisions composers make with respect to the choice of particular pitch materials or segmentation listeners and music theorists perform.[1] The present study aims at developing a method by which one can find what pc-sets and their combinations a composer of twelve-tone music prefers to others. A list of those pc-sets that a particular composer tends to choose over the others must be useful for music theorists because it serves as a guideline for segmentation and may well be a clue to find what Mead calls "syntax" of atonal pitch organization.[2]

Patterns of choices of pitch materials can be represented as a system of subjective probabilities.[3] Meyer (1967, 261) observes:

> Based upon innate patterns of action and perception, upon a complex array of cultural beliefs and attitudes, and upon learned experience, which depends partly upon frequency, listeners (including composers and performers) form complex systems of subjective probability feelings about musical events. Such internalized, subjective probability systems are the "beliefs"...

To find such "beliefs," music theorists often draw on psychological experiments that test subjects' preferences or expectations for some particular pitch materials. The present study will try another tack, however. Decision theory, the theory of rational decision making or rational choice originating from the eighteenth-century English clergyman Thomas Bayes, has developed methods by which one can deduce an agent's desirability scale, or a preference ranking, of the consequences that result from the agent's acts. The present paper uses decision theory to examine compositional decisions with respect to the acts of choosing pitch materials in twelve-tone music and tries to find what Mead (1989) calls the syntax of atonal pitch organization.[4] A great advantage of using twelve-tone music as a source of studies in atonal pitch organization is that, because composers of twelve-tone music use neither all the available twelve-tone row forms nor all the available subsets of a row form or a hexachord, one can find not only what pitch materials they chose but also what they avoided, that is, what they prefer over others.[5]

Twelve-tone techniques allow a composer to mechanically derive the definite number of row forms from a prime row step by step following a set of rules a finite number of times. Still, a composer has to make numerous compositional decisions when choosing some row forms from all the available ones, arranging them in certain orders, segmenting them into pc-sets, combining them into harmonies, and so on.[6]

For example, Stravinsky chose the following two twelve-tone rows for *Requiem Canticles* ("0" is always assigned to pitch-class C throughout the paper.):

	P1a	P1b
Prime row 1 (P1):	<<50B9A2>	<138647>>
T_n/T_nI:	[012358]	[012457]
I-vector:	<333231>	<333231>
Forte name:	6-Z40	6-Z11

	P2a	P2b
Prime row 2 (P2):	<<573461>	<B0298A>>
T_n/T_nI:	[012346]	[012346]
I-vector:	<443211>	<443211>
Forte name:	6-2	6-2

Example 1. The Twelve-Tone Rows Chosen for *Requiem Canticles*

At most ${}_6C_4 = 6!/(6-4)!4! = 15$ different tetrachords can be derived from a single hexachord.[7] Most T_n/T_nI hexachords, however, produce more than one tetrachord of the same T_n/T_nI type so that the total number of distinct T_n/T_nI tetrachords is not always 15. Derived from P1a: <50B9A2>, or 6-Z40: [012358], are the following 15 tetrachords:

Subset:	{0123}	{0125}	{0135}	{0235}	{1235}	{0128}
T_n/T_nI:	[0123]	[0125]	[0135]	[0235]	[0124]	[0126]
Name:	4-1	4-4	4-11	4-10	4-2	4-5

Subset:	{0138}	{0238}	{1238}	{0158}	{0258}	{1258}
T_n/T_nI:	[0237]	[0137]	[0127]	[0158]	[0258]	[0147]
Name:	4-14	4-Z29	4-6	4-20	4-27	4-18

Subset:	{0358}	{1358}	{2358}
T_n/T_nI:	[0358]	[0247]	[0136]
Name:	4-26	4-22	4-13

Example 2. The Tetrachords Derived from P1a, or 6-Z40

None of these T_n/T_nI types occurs more than once. So the relative frequencies ("R_Frq" for short) of the T_n/T_nI types are the same, 1/15:

Name	Frq	R_Frq	Name	Frq	R_Frq
4-1	1	1/15	4-6	1	1/15
4-4	1	1/15	4-20	1	1/15
4-11	1	1/15	4-27	1	1/15
4-10	1	1/15	4-18	1	1/15
4-2	1	1/15	4-26	1	1/15
4-5	1	1/15	4-22	1	1/15
4-14	1	1/15	4-13	1	1/15
4-Z29	1	1/15			
			Total:	15	

Example 3. The Relative Frequencies of the Tetrachordal Subsets

Since all the tetrachords have the same relative frequency, if randomly chosen, each of them has the same probability. The T_n/T_nI tetrachords Stravinsky derived from P1a for "Libera me" from *Requiem Canticles* do not have the same frequency, however. The frequency distribution of the T_n/T_nI tetrachords in "Libera me" is the following:[8]

Hexachord	Name	T_n/T_nI	Frq
P1a	4-5	[0126]	1
P1a	4-10	[0235]	1
P1a	4-11	[0135]	1
P1a	4-20	[0158]	1
P1a	4-22	[0247]	4
P1a	4-Z29	[0137]	3
P1a	4-13	[0136]	3
		Subtotal:	14
P1b	4-13	[0136]	1
P1b	4-Z15	[0146]	1
P1b	4-23	[0257]	1
		Subtotal:	3
		Total:	17

Example 4. Frequency Distribution of the Tetrachords in “Libera me”

Since 14 tetrachords are derived from 6-Z40 and the expected number of the frequency of each tetrachord is 14/15, the x^2 test (p=0.025) shows that Stravinsky’s choice of the tetrachordal subsets of 6-Z40 is biased, or intentional, at the 2.5% significance level.[9] It follows that Stravinsky did not choose the tetrachords at random but preferred some, in particular, 4-22, 4-Z29, and 4-13, over the others.

In addition to the “biased” choice of those tetrachords, several “arbitrary” alterations, omissions, and repetitions of some pitches are found not only throughout the entire repertoire of Stravinsky’s twelve-tone music but also in other composers such as Schoenberg. In the course of composition, therefore, those composers must have made some deliberate compositional decisions according to some constraints, or implicit principles of pitch organization, which resulted in preferences for particular pc-sets and “arbitrary” alterations. The theoretical studies that have appeared to date, however, hardly reveal the principles.[10]

I shall formulate compositional constraints as a system of subjective probability, show that preferences to compositional choices of particular pitch-materials are closely related to their probabilities, and deduce Stravinsky’s preference ranking of pitch materials. The examination of the patterns of his choice of pitch materials is expected to lead to the formulation of general principles of pitch organization in some atonal music.

In statistical terms, the approach outlined above is described as follows: The entire collection of twelve-tone rows and ordered and unordered pc-sets constitutes the “universe,” from which some “samples,” namely the pitch materials chosen by Stravinsky, are drawn in terms of some “marks” that are the common properties of the samples. Although Stravinsky chose only twenty twelve-tone rows and a limited number of pc-sets, that is, we have only a limited number of samples, by identifying those properties shared by all of them, it will be possible to “estimate” all pitch materials which Stravinsky might have chosen.

The choices of a twelve-tone rows and a particular operation of transposition, inversion, retrograde, retrograde-inversion, or rotation are dependent on each other. For example, as van den Toorn (1983, 390) points out, Stravinsky tends to concatenate those row forms which share the first and the last pitch classes. In other words, the construction of a probabilistic system of compositional choice has to take into account conditional probability and the ways compositional decisions depend on each other may vary from one piece or a section in a piece to another.[11] However, dependencies among compositional choices are so complex that it is a basic assumption throughout the present paper that each compositional decision is an independent act. Despite the relatively small size of Stravinsky’s twelve-tone repertoire, in order to fully explain how pitches are organized in his music, we still needs to examine many aspects of pitch organization such as relations to rhythm, contours, and instrumentation. It is, however, beyond the scope of the present study to consider all of those aspects.

AN OVERVIEW OF PITCH ORGANIZATION IN “Libera me”

In the following three sections, I shall try to find a probability system which is able to generate the harmonies in “Libera me” from *Requiem Canticles*. I have chosen “Libera me” as a source of this preliminary probabilistic model for the following three reasons: First, because “Libera me” consists mostly of tetrachordal harmonies, or four-note “chords,” the identification of pc-sets as pitch simultaneities seems less susceptible to complex issues of segmentation. Secondly, while, in those pieces where twelve-tone rows or hexachords are treated as lines, adjacent rows or hexachords tend to share pitch classes as links between them, in “Libera me,” by contrast, a common pitch class as a link between two hexachords appears only once, that is, {D} in m. 287. Therefore, the determining factor according to which Stravinsky chose the tetrachords may be something more harmonic rather than melodic. Thirdly, since it seems that the order of pitch classes in the hexachords did not concern Stravinsky in composing “Libera me,” the dependencies between compositional decisions are much less complex.

Spies (1967, 119) points out:

> As in all of Stravinsky’s larger recent works, there are some idiosyncratic practices which make 12- (or six-) counting extremely difficult. Instances of such usage are more often to be found in chords whose pitch content does not correspond partially or entirely to any of the available hexachord types, and whose identity amid the serial apparatus could be open, at least theoretically, to question.

For this reason, I assume that chords are one of the most “idiosyncratic” to Stravinsky’s atonal pitch organization and thus can be a clue to uncover some principles of his pitch organization.

Hexachords P1a and P1b generate all the ordered hexachords used for “Libera me.” All the ordered hexachords derived and chosen by Stravinsky can be uniquely identified, with two exceptions. That is, the first pc-set in m. 266, {70A2}, can be derived from T_0IP1a(I2): <70132A>, T_6P1b(P_6): <7920A1>, or “vertical” I_9: {2B07A7} made of the twelve-tone row. Since all the rest of pc-sets in “Libera me” do not contain any “verticals” with another possible exception, namely vertical I_7: {9288B0} in mm. 276–77, it must be either T_0IP1a(I_2) or T_6P1b(P_6). I choose the former because all the other pc-sets except for the last two in “Libera me” are derived from T_nIP1’s. There are also two alternatives with respect to the derivations of the hexachords in mm. 285–86 and in the first half of m. 287, namely T_5P1a: <A54237> and T_3P1a: <A83574>. I guess that Stravinsky took the former so that {D} in m. 287 is shared by the consecutive two hexachords.

The ordered hexachords appear in the following way:[12]

266 T_0IP1a(I2):<70132A>	$T_{11} :\to$	267–69 $T_{11}I$P1a(I1):<6B0219> $T_4 :\to$
270–72 T_3IP1a(I5):<A34651>	$\to$	272–73 T_1IP1b(I11):<0A5796> $\to$
274–75 T_2IP1a(I4):<923540>	$\to$	276–77 T_3IP1b(I1):<2079B8> $\to$
278–79 T_3IP1a(I5):<A34651>	$T_7 :\to$	280–84 $T_{10}I$P1a(I0):<5AB108> $T_3I :\to$
285–97 T_5P1a(P5):<A54237>	$T_{10} :\to$	287–88 T_3P1a(P3):<832015>

Example 5. The Order and the Transformation of the Hexachords

Pc-sets identified as tetrachordal and trichordal “chords” appear as shown in Example 43. Each syllable is assigned to a single chord and every phrase of the words except for the last two has similar rhythmic patterns consisting of repeated quarter-notes coupled with a half note and a whole note. Most hexachords occupy two measures. T_3IP1a(I5): <A34651> and T_1IP1b(I11): <0A5796> interlock in m. 272 so that pc-set {649} occurs. {7} is omitted from T_3Ib1(I1b1): <2079B8> in mm. 276–77. Mm. 278–79 are the same as mm. 270–71, as the words are repeated.

It might seem that consecutive pitch classes in an ordered hexachord tend to be chosen to form unordered pc-sets, or chords. In other words, particular orders of pitch classes in ordered hexachords might have affected Stravinsky's choice. T_5P1a(P5): <A54237> in mm. 285–87, for example, might be ordered hexachord r_3RT_5P1a: <45A732>. Also the hexachord at the beginning of "Libera me" in m. 266 might be r_2RT_0IP1a: <07A231>, because {07} precedes {07A2}.

When adjacent four pitch classes in P1a are mechanically chosen, the following 6 tetrachords will be derived:

Subset:	{0123}	{1235}	{2358}	{3580}	{5801}	{8012}
T_n/T_nI:	[0123]	[0124]	[0136]	[0358]	[0158]	[0126]
Name:	4-1	4-2	4-13	4-26	4-20	4-5

Example 6. PC-Sets of Adjacent Four Pitch Classes from P1a

Among these, 4-13, 4-20, and 4-5 are chosen for "Libera me." Although Stravinsky chose 4-13 three times, he chose 4-Z29 also three times and 4-22 four times, both of which do not consist of consecutive pitch classes in P1a. As shown below, 14 tetrachords from P1a appear in "Libera me." So, only 5 out of 14 tetrachords retain the order of pitch classes in their ordered superset.

From P1b are derived the following:

Subset:	{0124}	{1245}	{2457}	{4570}	{5701}	{7012}
T_n/T_nI:	[0124]	[0134]	[0235]	[0237]	[0157]	[0127]
Name:	4-2	4-3	4-10	4-14	4-16	4-6

Example 7. PC-Sets of Adjacent Four Pitch Classes from P1b

Stravinsky chose none of these tetrachords. Thus it seems that the order of pitch classes in P1 has little to do with determining the derivations of the tetrachords.

In sum, the chords in "Libera me" are derived as follows:

Hexachord	Chord	T_n/T_nI	Name
T_3P1a(T3):$< 832015 >$	{1358}	[0247]	4-22
T_5P1a(T5):$< A54237 >$	{469}	[025]	3-7
T_5P1a(T5):$< A54237 >$	{457A}	[0136]	4-13
T_0IP1a (I2):$< 70132A >$	{027A}	[0247]	4-22
T_2IP1a (I4):$< 923540 >$	{0459}	[0158]	4-20
	{0239}	[0136]	4-13
T_3IP1a (I5):$< A34651 >$	{135A}	[0247]	4-22
	{346A}	[0137]	4-Z29
$T_{10}I$P1a(I0):$< 5AB108 >$	{15AB}	[0137]	4-Z29
	{018A}	[0135]	4-11
	{18AB}	[0235]	4-10
	{015B}	[0126]	4-5
$T_{11}I$P1a(I1):$< 6B0219 >$	{129}	[015]	3-4
$T_{11}I$P1a(I1):$< 6B0219 >$	{069B}	[0136]	4-13
T_1IP1b (I11):$< 0A5796 >$	{057A}	[0257]	4-23
T_3IP1b (I1):$< 2079B8 >$	{289B}	[0136]	4-13
	{0289}	[0146]	4-Z15

Example 8. All the Hexachords and Derived Tetrachords

There can be several different levels of generalization in terms of identification of groups of pitch materials such as pitch-sets, T_n-types, and T_n/T_nI types. Some theorists such as Richmond Browne (1974) are critical of regarding T_n/T_nI types as perceivable pitch materials in some contexts.[13] For now, however, I would like to focus on the most generalized ones, namely T_n/T_nI types.

CHOOSING TETRACHORDS FROM HEXACHORDS IN "Libera me"

Next, let us find the probabilities of Stravinsky's choices of particular pc-sets for "Libera me" and the patternings of his compositional decisions by means of the probabilities. Since all the tetrachords have the same relative frequency, if randomly chosen, each of them has the same probability. However, the T_n/T_nI tetrachords Stravinsky derived from P1a for "Libera me" do not have the same frequency. The relative frequencies of the T_n/T_nI tetrachords appeared in the piece are the following:

Name	Frq	R_Frq	Name	Frq	R_Frq
4-1	0	0/14=0.0000	4-6	0	0/14=0.0000
4-4	0	0/14=0.0000	4-20	1	1/14=0.0714
4-11	1	1/14=0.0714	4-27	0	0/14=0.0000
4-10	1	1/14=0.0714	4-18	0	0/14=0.0000
4-2	0	0/14=0.0000	4-26	0	0/14=0.0000
4-5	1	1/14=0.0714	4-22	4	4/14=0.2857
4-14	0	0/14=0.0000	4-13	3	3/14=0.2143
4-Z29	3	3/14=0.2143			

Example 9. Relative Frequencies of the Chosen Tetrachords

The relative frequencies here might amount to the likelihood of occurrences of tetrachords. In other words, it could be said that 4-22, for instance, is more likely to occur than 4-2 because the relative frequency of the former is greater than that of the latter. They might also correspond to Stravinsky's preferences for particular pc-sets. I shall return to these points later.

Likewise, the following tetrachords come from P1b: <138647>, or 6-Z11:

Subset:	{0124}	{0125}	{0145}	{0245}	{1245}	{0127}
T_n/T_nI:	[0124]	[0125]	[0145]	[0135]	[0134]	[0127]
Name:	4-2	4-4	4-7	4-11	4-3	4-6
Subset:	{0147}	{0247}	{1247}	{0157}	{0257}	{1257}
T_n/T_nI:	[0147]	[0247]	[0136]	[0157]	[0257]	[0146]
Name:	4-18	4-22	4-13	4-16	4-23	4-Z15
Subset:	{0457}	{1457}	{2457}			
T_n/T_nI:	[0237]	[0236]	[0235]			
Name:	4-14	4-12	4-10			

Example 10. The Tetrachords Derived from P1b, or 6-Z11

No tetrachord type occurs more than once, either. The relative frequencies of the T_n/T_nI tetrachords are, therefore, as follows:

Name	Frq	R_Frq	Name	Frq	R_Frq
4-2	1	1/15	4-13	1	1/15
4-4	1	1/15	4-16	1	1/15
4-7	1	1/15	4-23	1	1/15
4-11	1	1/15	4-Z15	1	1/15
4-3	1	1/15	4-14	1	1/15
4-6	1	1/15	4-12	1	1/15
4-18	1	1/15	4-10	1	1/15
4-22	1	1/15			

Example 11. The Relative Frequencies of the Tetrachordal Subsets

Name	Frq	S_Prb	Name	Frq	S_Prb
4-2	0	0/3=0.0000	4-13	1	1/3=0.3333
4-4	0	0/3=0.0000	4-16	0	0/3=0.0000
4-7	0	0/3=0.0000	4-23	1	1/3=0.3333
4-11	0	0/3=0.0000	4-Z15	1	1/3=0.3333
4-3	0	0/3=0.0000	4-14	0	0/3=0.0000
4-6	0	0/3=0.0000	4-12	0	0/3=0.0000
4-18	0	0/3=0.0000	4-10	0	0/3=0.0000
4-22	0	0/3=0.0000			

Example 12. Relative Frequencies of the Chosen Tetrachords

In "Libera me," 4-13, 4-Z15, and 4-23 are chosen once each from P1b. Thus the relative frequencies of the tetrachords in the piece might be those in Example 12.

However, if Stravinsky had chosen the twelve-tone row and the pair of hexachords first and then derived the tetrachords from them, it would be more natural to assume that, when he composed "Libera me," both P1a and P1b and all the tetrachords derived from them were equally available to him. From both P1a and P1b are derived the following 21 distinct T_n/T_nI tetrachords:

Name	Frq	R_Frq	Name	Frq	R_Frq
4-1	1	1/30	4-14	2	2/30
4-2	2	2/30	4-Z15	1	1/30
4-3	1	1/30	4-16	1	1/30
4-4	2	2/30	4-18	2	2/30
4-5	1	1/30	4-20	1	1/30
4-6	2	2/30	4-22	2	2/30
4-7	1	1/30	4-23	1	1/30
4-10	2	2/30	4-26	1	1/30
4-11	2	2/30	4-27	1	1/30
4-12	1	1/30	4-Z29	1	1/30
4-13	2	2/30			
			Total:	30	

Example 13. All the T_n/T_nI Tetrachords Derived from both P1a and P1b

The two hexachords provided Stravinsky with 30 tetrachords or 21 distinct ones, from which he chose 9 distinct T_n/T_nI types for "Libera me." There are altogether 29 distinct T_n/T_nI tetrachords. So, at the stage where Stravinsky chose the prime row, the number of available distinct T_n/T_nI tetrachords decreased from 29 to 21.

COMPUTING THE PROBABILITIES OF THE TETRACHORDS

Although 9 distinct T_n/T_nI tetrachords are chosen for "Libera me," since some appear more than once, total 17 tetrachords are employed in the composition. It could be said, therefore, that Stravinsky chose those 17 tetrachords one by one, that is, 17 times, out of the 21 distinct T_n/T_nI tetrachords available for him. Recall, however, that it is a basic premise throughout the present paper that each compositional decision is an independent act. In other words, when Stravinsky *randomly* chose those 17 tetrachords one by one, each trial of choosing a T_n/T_nI tetrachord from the hexachords was a Bernoulli trial, that is, an event that one chooses, say, 4-22 either takes place or not and does not depend on any previous event, and the number of times the event occurs follows a binomial distribution $B(n, p)$, where n is the number of trials, or acts of choosing the tetrachord, and p its probability. Thus the size of the repository of tetrachords available here is not 21 but 30.

Accordingly, since the discrete density function of (n, p) is

$$f(x) =_n C_x p^x q^{(n-x)}$$

and the mean of x, $E(x)$, of $B(n,p)$ is

$$E(x) = np,$$

the probability of choosing 4-1, for example, is, since it is not chosen for "Libera me,"

$${}_{17}C_0 \left(\tfrac{1}{30}\right)^0 \left(\tfrac{29}{30}\right)^{17} = 0.561959$$

and that of choosing 4-13 is, since it is chosen 4 times,

$${}_{17}C_4 \left(\tfrac{2}{30}\right)^4 \left(\tfrac{28}{30}\right)^{13} = 0.019173.$$

The probability distributions of $B(17, 1/30)$ with respect to a random variable x, which denotes the number of times a particular tetrachord is chosen at 17 times of trials, is as follows:

x:	0	1	2	3	4	5
Probability:	0.561959	0.329424	0.090876	0.015668	0.001891	0.000170

Example 14. Binomial Distribution of Random Variable x in $B(17, 1/30)$

And that of $B(17, 2/30)$ is:

x:	0	1	2	3	4	5
Probability:	0.309475	0.375791	0.214738	0.076692	0.019173	0.003561

Example 15. Binomial Distribution of Random Variable x in $B(17, 2/30)$

It would seem natural that the more a particular tetrachord is chosen, the less its probability becomes.

If $B(17, 1/30)$, then $E(x) = 17/30 = 0.566667$. Thus, if Stravinsky had chosen 17 T_n/T_nI tetrachords from the repository of the 30 tetrachords at random, those tetrachords of which relative frequency is 1/30 are expected to occur 0.566667 times and those of which relative frequency is 2/30 occur $17(2/30) = 1.13333$ times. Since the probability of choosing a particular T_n/T_nI tetrachord such as 4-22 4 times is as low as 0.019173, approximately once every 50 trials, we can conclude that Stravinsky did not randomly choose those 17 T_n/T_nI tetrachords but preferred some particular ones and the low probabilities of some tetrachords such as 4-22 reflect his strong preference for them.

Some consideration of the number of tetrachords may be necessary here. Because mm. 278 and 279 are an exact repetition of mm. 270 and 271 as words "Quando coeli movendi sunt et terra" are repeated, the two tetrachords, 4-22 and 4-Z29, in mm. 270–71 should perhaps be excluded from the count. However, if Stravinsky had not liked those two tetrachords appearing again, he would have been able to remove the repeated words or change the pc-sets for the words. Considering his flexible adoption of the Requiem setting, I would like to include those tetrachords.[14]

Consequently, the probabilities of all the 21 T_n/T_nI tetrachords are as follows:

Name	R_Frq	Frq	$B(n,p)$	Name	R_Frq	Frq	$B(n,p)$
4-1	1/30	0	0.561959	4-14	2/30	0	0.309475
4-2	2/30	0	0.309475	4-Z15	1/30	1	0.329424
4-3	1/30	0	0.561959	4-16	1/30	0	0.561959
4-4	2/30	0	0.309475	4-18	2/30	0	0.309475
4-5	1/30	1	0.329424	4-20	1/30	1	0.329424
4-6	2/30	0	0.309475	4-22	2/30	4	0.019173
4-7	1/30	0	0.561959	4-23	1/30	1	0.329424
4-10	2/30	1	0.375791	4-26	1/30	0	0.561959
4-11	2/30	1	0.375791	4-27	1/30	0	0.561959
4-12	1/30	0	0.561995	4-Z29	1/30	3	0.015668
4-13	2/30	4	0.019173				
					Total:	17	

Example 16. Probabilities of Choosing T_n/T_nI Tetrachords

Here "R_Frq" denotes a relative frequency to all the 30 T_n/T_nI tetrachords derived from both P1a and P1b and "Frq," the frequency of the occurrence in "Libera me."

BAYESIAN PRINCIPLES OF CHOICE

Now, let us examine formal decision-making in the framework of Bayesian decision theory and consider its application to atonal pitch organization so that we can compute Stravinskys desirabilities, or preference ranking, of the T_n/T_nI tetrachords. Current decision theory demonstrates that, when probabilities of conditions and preferences among the consequences of acts under the conditions are available, we can calculate the desirabilities for the consequences and construct a desirability scale that shows an exact preference ranking among the acts and, when desirabilities for consequences are available, we can deduce their probabilities.

A problem of decision making is presented in the following three matrices:

	Condition 1	Condition 2
Act 1	Consequence 11	Consequence 12
Act 2	Consequence 21	Consequence 22

Example 17. Consequence Matrix

"Condition 1" and "Condition 2" are incompatible conditions under which the agent choose either "Act 1" or "Act 2." "Consequence" is the consequence of each Act under each Condition.

	Condition 1	Condition 2
Act 1	Desirability 11	Desirability 12
Act 2	Desirability 21	Desirability 22

Example 18. Desirability Matrix

"Desirability" is the numerical value of the agent's desirability assigned to each Consequence.

	Condition 1	Condition 2
Act 1	Probability 11	Probability 12
Act 2	Probability 21	Probability 22

Example 19. Probability Matrix

"Probability" is the numerical value of the probability the agent assigns to each Condition.

In these circumstances, if

$$Estimated\_Desirability1 = Des11 \cdot Prob11 + Des12 \cdot Prob12$$
$$Estimated\_Desirability2 = Des21 \cdot Prob21 + Des22 \cdot Prob22$$

("Probability" is abbreviated as "Prob" and "Desirability," "Des.") and Estimated_Desirability1 is greater than Estimated_Desirability2, the desirable choice is Act 1 because the Bayesian principle tells us:

> . . . *choose an act of maximum estimated desirability.* (*An* act rather than *the* act, since two or more of the possible acts may have the same, maximum estimated desirability.) (Jeffrey 1983, 1)

For example, consider famous "Pascal's wager," formulated by Jeffrey (1983 12–13) as follows:

	God exists	There is no God
Succeed in believing	Eternal life	Finite life, deluded
Remain an atheist	A bad situation	The presumed status

Example 20. Consequence Matrix of Pascal's Wager

Suppose that the agent, who is trying to believe in God, has the following desirabilities for these consequences:

	God exists	There is no God
Succeed in believing	$INFINITE$	x
Remain an atheist	z	y

Example 21. Desirability Matrix of Pascal's Wager

where $INFINITE$ is the infinite desirability, x, y, and z are finite. Note that these assignments are based on the agent's subjective desirabilities. People like, say, Woody Allen for whom life is horrible and miserable, would assign, instead of $INFINITE$, a finite or even negative value to "Eternal life." If the agent assigns probabilities, which are the degrees of the agent's beliefs, to the conditions as follows:

	God exists	There is no God
Succeed in believing	$1/1,000,000$	$999,999/1,000,000$
Remain an atheist	$1/1,000,000$	$999,999/1,000,000$

Example 22. Probability Matrix of Pascal's Wager

the agent obtains estimated desirabilities as follows:

Succeed in believing: $INFINITE(1/1,000,000) + x(999,999/1,000,000)$
Remain an atheist: $z(1/1,000,000) + y(999,999/1,000,000)$

Since z and y are finite, the estimated desirability of the act "Succeed in believing" is greater than that of "Remain an atheist" and thus, according to Bayesian principle, the former is the choice the agent should take.

It is also possible to deduce an agent's desirabilities from probabilities when certain conditions are satisfied, as Jeffrey (1983, 44) explains:

> The general problem of measuring desirabilities has this form. We are presented with a preference ordering in which a consequence C is ranked between two given consequences, A and B.
>
> B
> C
> A
>
> We seek to find exactly where C lies in the desirability interval from A to B. We can do this if a deliberation can be found in which the consequence matrix is as follows; in which the probabilities of the two conditions are known and independent of which act is performed; and in which the agent is indifferent between the two acts.

The next example, taken from Jeffrey (1983, 11–12, 42–44), illustrates how to solve problems like this. According to the American Cancer Society, the percentages of American men aged 35 expected to die before age 65 are the following:

Nonsmokers	23%
Cigar and pipe smokers	25%
Cigarette smokers:	
Less than 1/2 pack a day	27%
1/2 to 1 pack a day	34%
1 to 2 packs a day	38%
2 or more packs a day	41%

The acts the agent can choose are the following:

C: Continue to smoke 2 or more packs of cigarettes a day
S: Switch from cigarettes to pipes and cigars

The relevant conditions for these acts are:

D: Die before the age of 65
L: Live to age 65 or more

The agent assigns desirabilities to the consequences as follows:

	D	L
C	$d+c$	$d+c+l$
S	d	$d+l$

Example 23. Desirability Matrix of a Smoker

where d is the agent's lowest desirability, l, the agent's preference for living to age 65 or more over dying before age 65, and c, the agent's preference for cigarette over pipes and cigars.

	D	L
C	.41	.59
S	.25	.75

Example 24. Probability Matrix of a Smoker

Then, by subtracting d from the desirability matrix, it will be:

	D	L
C	c	$c+l$
S	0	l

Example 25. Desirability Matrix subtracted d

The estimated desirabilities will be, therefore:

$C : .41c + .59(c+l) = c + .59l$
$S : .75l.$

If the agent is indifferent between the two acts, in other words, if the agent likes the two acts equally well, then:

$C = S$
$c + .59l = .75l$
$c = .16l.$

The desirability matrix will be:

	D	L
C	$.16l$	$1.16l$
S	0	l

Since l is not 0, by dividing each entry by l, we have:

	D	L
C	.16	1.16
S	0	1

Example 26. Desirability Matrix of a Smoker

Now we know the agent's desirability for each consequence on a desirability scale:

1.16 = Live to 65 or more, smoking 2 or more packs or cigarettes a day.
1.00 = Live to 65 or more, smoking only pipes and cigars.
0.16 = Die before 65, smoking 2 or more packs of cigarettes a day.
0.00 = Die before 65 in spite of smoking only pipes and cigars.

Example 27. Desirability Scale of a Smoker

It should be clear that, only when an agent finds oneself indifferent between the two acts and the probabilities of conditions are not affected by a particular choice of an act, we can deduce desirabilities from probabilities. In other words, if we can come up with two or more acts with the same estimated desirability and are provided with probabilities, we can obtain the agent's desirability for each consequence and hence a desirability scale, or preference ranking. Ramsey's theory (Ramsey 1930/1950) and the von Neumann-Morgenstern Method (von Neumann and Morgenstern 1944) show us, when necessary, how to devise such acts, discover an agent's preference ranking, and calibrate its desirability scale as finely as need be. The way probabilities and preference ranking are related will be explained as the discussion goes.[15]

NOTE ON THE NOTIONS OF PROBABILITY

Although a detailed discussion is beyond the scope of the present paper, I would like to discuss here the notion of probability because there is much confusion about it.[16] A modern theory of probability originates in the correspondence between Blaise Pascal and Pierre de Fermat in 1654. Since then many philosophers and mathematicians have contributed to the theory. Especially, *Logique de Port-Royal* (1662), Jacques Bernouilli's (1654–1750) *Ars conjectandi*(1713), Thomas Bayes (1671–1746), Pierre Simon Laplace's (1749–1827) *Theorie analytique des probabilities*(1812) are among the most important.

The notion of probability itself had not been examined until the 19th century, however. In order to illuminate the notion of subjective probability and make clear why a Bayesian point of view is appropriate for the current study in compositional decision, I would like to discuss the differences among notions of probability. As Kyburg and Smokler (1980, 4) observe:

> To take probability as simply a mathematical function of a certain sort is to take it as an undefined term in a formal system; but when we come to apply the formal system to the world, when we begin to talk about the probability of certain specific events or even of certain kinds of events, . . . we are driven to think more closely about the notion of probability itself. We must find some connection between this abstract entity which satisfies certain mathematical stipulations and the pragmatic content . . . of the important statements . . .

Thus we need to make such a connection in some way, or an interpretation of the notion of probability. Three types of interpretations have so far been proposed, namely "frequency theory," "logical probability," and "subjective probability."

According to the frequency, empirical, or objective interpretation, "probability" is nothing more than the ratio, or the relative frequency, of the number of equiprobable events to the total number of equiprobable ones. This interpretation has been championed by, among others, Richard von Mises and Hans Reichenbach. From this standpoint, however, it would be difficult to interpret such a proposition or hypothesis as "the probability that it will rain tomorrow is 90%," for it cannot be a ratio.

In this regard, logical probability deals with the relation between a hypothesis and evidence. This interpretation of "probability" has been supported by John Maynard Keynes, Frank Plumpton Ramsey, and Rudolf Carnap. According to Carnap (1962), a "probability" represents the logical relationship between a hypothetical statement and a statement that refers to the evidence. In other words, a probability is the "degree of confirmation," $c(h, e)$, of hypothesis h which is confirmed by evidence (a statement representing the result of an experiment or an observation) e.

Lastly, Jeffrey (1983, 60) characterizes subjective probability as follows:

> Subjective probability is partial belief. To say that an agent attributes (subjective) probability .7 to the proposition that it will rain tomorrow is to say that his degree of belief in that proposition is .7; and this is in turn means (roughly) that he would be just willing to pay $.70 in order to receive $1 if it rains tomorrow and nothing if it doesn't.

As stated earlier, I assume that, when Stravinsky made compositional decisions, he was doing "gambles" of this sort consulting his own "aesthetic" desire. It would be obvious that both frequency and logical interpretations are inappropriate with respect to conditions for acts of composing music.

COMPOSITIONAL CHOICE IN A BAYESIAN FRAMEWORK

By applying decision theory outlined in the previous section, it becomes possible for us to trace composers' compositional decisions in terms of their choice of pitch materials. In an attempt to find probabilities of pitch materials and their preference ranking in their twelve-tone music, I assume that, when composers compose twelve-tone music, they observe the Bayesian principle, that is, they choose an act of maximum estimated desirability at each step of composition so that it satisfies their "aesthetic desire." In my discussion, therefore, "aesthetic desire" replaces "economic utility" by which Bayesian decision theories usually measure desirabilities. From the Bayesian point of view, therefore, Stravinsky chosen, for instance, particular tetrachords in "Libera me" so that he maximized his estimated aesthetic desirabilities. Based on this assumption, I shall try to deduce from probabilities his desirabilities for particular pitch organization.

When Stravinsky selected a tetrachordal harmony, for example, from an unordered hexachord, ${}_6C_4 = 6!/4!(6-4) = 15$ tetrachords were available to him. Those chosen from the 15 tetrachords by him for a particular composition must therefore have higher preferences over the others and possess higher positions in his preference ranking of tetrachords. Also, those pc-sets, some pitch classes of which are "arbitrarily" changed, must have higher positions on a preference ranking than those with the original pitch classes.

When Stravinsky derived a tetrachord from a hexachord, some property possessed by the tetrachord which satisfied some principle of pitch organization in his twelve-tone music might have been the determining factor in choice. Then, One form of a probability matrix, when the relevant condition to acts is some principle of pitch organization in terms of tetrachordal harmony, may be as follows:

	Satisfy a principle	Not satisfy a principle
Choose tetrachord 1	Probability 11	Probability 12
Choose tetrachord 2	Probability 21	Probability 22
⋮	⋮	⋮
Choose tetrachord n	Probability n1	Probability n2

Example 28. Probability Matrix of Choice of Tetrachords

If Stravinsky had chosen tetrachords based on the properties they possess, the matrix above could be modified as follows:

	Satisfy a principle	Not satisfy a principle
Choose Properties 1	Probability 11	Probability 12
Choose Properties 2	Probability 21	Probability 22
⋮	⋮	⋮
Choose Properties n	Probability n1	Probability n2

Example 29. Probability Matrix of Choice of Properties

A corresponding consequence matrix is the following:

	Satisfy a principle	Not satisfy a principle
Choose Properties 1	Tetrachord 11	Tetrachord 12
Choose Properties 2	Tetrachord 21	Tetrachord 22
⋮	⋮	⋮
Choose Properties n	Tetrachord n1	Tetrachord n2

Example 30. Consequence Matrix of Choice of Properties

If we can find exactly what properties these are, principles which prescribe Stravinsky's compositional decisions will be formulated as rules of choosing particular properties and we could explain why he chose particular tetrachords.

When different hexachords are available, because derivable tetrachords are confined by them, they are conditions under which Stravinsky made decisions on choosing some harmonic properties, the following matrix would represent such circumstances:

	Hexachord 1	Hexachord 2	...	Hexachord j
Properties 1	Tetrachord 11	Tetrachord 12	...	Tetrachord 1j
Properties 2	Tetrachord 21	Tetrachord 22	...	Tetrachord 2j
⋮	⋮	⋮	⋮	⋮
Properties i	Tetrachord i1	Tetrachord i2	...	Tetrachord ij

Example 31. Consequence Matrix of Choosing Properties

Accordingly, a corresponding probability matrix will be as follows:

	Hexachord 1	Hexachord 2	...	Hexachord j
Properties 1	Probability 11	Probability 12	...	Probability 1j
Properties 2	Probability 21	Probability 22	...	Probability 2j
⋮	⋮	⋮	⋮	⋮
Properties i	Probability i1	Probability i2	...	Probability ij

Example 32. Consequence Matrix of Choosing Properties

Also, as explained, the following probability matrix may represent possibilities of deriving a tetrachord from each hexachord:

	Hexachord 1	Hexachord 2	...	Hexachord j
Tetrachord 1	Probability 11	Probability 12	...	Probability 1j
Tetrachord 2	Probability 21	Probability 22	...	Probability 2j
⋮	⋮	⋮	⋮	⋮
Tetrachord i	Probability i1	Probability i2	...	Probability ij

Example 33. Probability Matrix of Choosing Tetrachords

The corresponding desirability matrix will be as follows:

	Hexachord 1	Hexachord 2	...	Hexachord j
Tetrachord 1	Desirability 11	Desirability 12	...	Desirability 1j
Tetrachord 2	Desirability 21	Desirability 22	...	Desirability 2j
⋮	⋮	⋮	⋮	⋮
Tetrachord i	Desirability i1	Desirability i2	...	Desirability ij

Example 34. Desirability Matrix of Choices of Tetrachords

Next, I shall construct some of these types of matrices and find Stravinsky's desirability scale of choosing tetrachords. We shall need other probability matrices and corresponding desirability matrices later.

COMPUTING THE DESIRABILITIES OF THE TETRACHORDS

Now, based on the binomial distributions calculated earlier, we can compute Stravinsky's desirability of each T_n/T_nI tetrachord. As suggested earlier, one possible form of a probability matrix with respect to acts of choosing the tetrachords may be as follows:

	Satisfy principles	Not satisfy principles
Choose 4-Z29 3 times	0.015668	0.984332
Choose 4-13 4 times	0.019173	0.980827
Choose 4-5 once	0.329424	0.670576
Choose 4-10 once	0.375791	0.624209
Choose ¬4-1	0.561959	0.438041
Choose ¬4-2	0.309475	0.690525

Example 35. Probability Matrix of Choosing the Tetrachords

Here, ¬4-1 stands for no 4-1 is chosen. Then, a corresponding desirability matrix could be as follows:

	Satisfy principles	Not satisfy principles
Choose 4-Z29 3 times	$a+b+c+d+e+f$	$a+b+c+d+e$
Choose 4-13 4 times	$b+c+d+e+f$	$b+c+d+e$
Choose 4-5 once	$c+d+e+f$	$c+d+e$
Choose 4-10 once	$d+e+f$	$d+e$
Choose ¬4-1	$e+f$	e
Choose ¬4-2	f	0

Example 36. Desirability Matrix of Choosing the Tetrachords

Suppose that Stravinsky likes 4-1 happening no times as much as 4-22 four times. Since the probability matrix above is derived from Stravinsky's piece, we could assume that he likes 4-Z29 being chosen with probability 0.015668 as much as 4-10 being chosen with probability 0.375791, 4-1 being not chosen with probability 0.561959, and so on (Remember that the probabilities assigned to 4-1 and 4-2, namely 0.561959 and 0.309475 respectively, are those of 4-1 and 4-2 being not chosen). In other words, the estimated desirability for each tetrachord would be the same. Then, we obtain the following equation:

$$\begin{aligned}
&0.015668(a+b+c+d+e+f)+0.984332(a+b+c+d+e)=\\
&0.019173(b+c+d+e+f)+0.980827(b+c+d+e)=\\
&0.329424(c+d+e+f)+0.670576(c+d+e)=\\
&0.375791(d+e+f)+0.624209(d+e)=\\
&0.561959(e+f)+0.438041e=0.309475f
\end{aligned}$$

By solving this equation, we get the following desirabilities:

$$\begin{aligned}
&d=0.186168f\\
&e=-0.252484f\\
&e+f=0.747516f\\
&d+e=-0.066316f\\
&d+e+f=0.933684f\\
&c+d+e=-0.019949f\\
&c+d+e+f=0.980051f\\
&b+c+d+e=0.290302f\\
&b+c+d+e+f=1.290302f\\
&a+b+c+d+e=0.293807f\\
&a+b+c+d+e+f=1.293807f
\end{aligned}$$

Then, the desirability matrix will be as follows:

	Satisfy principles	Not satisfy principles
Choose 4-Z29 3 times	$1.293807f$	$0.293807f$
Choose 4-13 4 times	$1.290302f$	$0.290302f$
Choose 4-5 once	$0.980051f$	$-0.019949f$
Choose 4-10 once	$0.933684f$	$-0.066316f$
Choose ¬4-1	$0.747516f$	$-0.252484f$
Choose ¬4-2	f	0.000000

Example 37. Desirability Matrix

By dividing every entry by f, we obtain the following:

	Satisfy principles	Not satisfy principles
Choose 4-Z29 3 times	1.293807	0.293807
Choose 4-13 4 times	1.290302	0.290302
Choose 4-5 once	0.980051	-0.019949
Choose 4-10 once	0.933684	-0.066316
Choose ¬4-1	0.747516	-0.252484
Choose ¬4-2	1.000000	0.000000

Example 38. Desirability Matrix

The entries for 4-1 and 4-2 should be negative, for the probabilities of them are those of their not taking place. Therefore, we obtain the following desirabilities:

4-Z29	1.293807
4-13	1.290302
4-5	0.980051
4-10	0.933684
¬4-1	0.747516
¬4-2	1.000000

Example 39. Desirabilities of PC-Sets

Consequently, we obtain the following desirability scale for all the tetrachords derived from P1:

1.293807	4-Z29
1.290302	4-13, 4-22
1.000000	¬4-2, ¬4-4, ¬4-6, ¬4-14, ¬4-18
0.980051	4-5, 4-Z15, 4-20, 4-23
0.933684	4-10, 4-11
0.747516	¬4-1, ¬4-3, ¬4-7, ¬4-12, ¬4-16, ¬4-26, ¬4-27

Example 40. Desirability Scale of Tetrachords

It might seem puzzling that, while the desirability of 4-Z29 is 1.293807, that of its Z-related complement 4-Z15 is 0.980051. If Stravinsky chose pc-sets according to their interval-class contents, Z-related pc-sets would have the same desirability. In fact, although some theorists claim that Z-related pc-sets appear in a related fashion in atonal compositions, it is not the case for the entire repertory of Stravinsky's twelve-tone music. One possible answer to this problem might be that he prefers 4-Z29 to 4-Z15 for a melodic reason. Apparently "Libera me" is not very melodic, however. Another explanation would be that, since Stravinsky chose hexachord P1a prior to P1b as a source of tetrachords, he chose 4-Z29 more than 4-Z15, which comes from P1b. It also explains why most tetrachords in "Libera me" belong to P1a. Indeed, two out of three tetrachords chosen from P1b, namely 4-Z15 and 4-23 cannot be derived from P1a. I shall provide one more explanation later.

One might wonder if the way of measuring preference for pc-sets I have employed is appropriate. Stravinsky's preferences for pc-sets have so far been measured only by means of their frequencies. However, the frequencies determine their probabilities, from which, in turn, desirabilities are derived. So a pc-set with a higher frequency does not necessarily possess a higher desirability than one with a lower frequency because the former can have a higher relative frequency, or be more likely to happen.[17]

If we interpret the event that Stravinsky chose 4-Z29, for instance, 4 times as such that he preferred a certain property possessed by 4-Z29 happening 4 times, the consequence matrix is represented as follows:

	Satisfy principles	Not satisfy principles
Properties 1	Choose 4-Z29 3 times	The other tetrachords
Properties 2	Choose 4-13 4 times	The other tetrachords
Properties 3	Choose 4-5 once	The other tetrachords
Properties 4	Choose 4-10 once	The other tetrachords
Properties 5	Choose no 4-1	The other tetrachords
Properties 6	Choose no 4-2	The other tetrachords

Example 41. Alternative Consequence Matrix of Choosing Properties

Accordingly, the desirability scale will be the following:[18]

Choice	Desirability	Consequence
Properties 1	1.293807	4-Z29 3 times
Properties 2	1.290302	4-13 or 4-22 4 times
Properties 6	1.000000	¬4-2, ¬4-4, ¬4-6, ¬4-14, or ¬4-18
Properties 3	0.980051	4-5, 4-Z15, 4-20, or 4-23 once
Properties 4	0.933684	4-10 or 4-11 once
Properties 5	0.747516	¬4-1, ¬4-3, ¬4-7, ¬4-12, ¬4-16, ¬4-26, or ¬4-27

Example 42. Stravinsky's Desirability Scale of Properties

Given this desirability scale, in order to explain why Stravinsky preferred 4-Z29, 4-13, and 4-22 but not 4-2, 4-4, and so on, we need to find what properties Properties 1 and 2 are, that is, exactly which properties among those possessed by those preferred tetrachords are the determining factors for Stravinsky to choose the tetrachords. I shall discuss this issue in the rest of the paper.

AN EXTENSION OF POSITION-FINDING TO ATONAL CONTEXTS

Those properties that are the determining factors by which Stravinsky chose the tetrachords for "Libera me" may be designated by using the notion POSITION FINDING (Browne 1981) and the "Intervallic Rivalry Model" (Butler 1989, 1992).[19] Browne argues that, because of the unique multiplicity property of the interval vector of the diatonic set, when we hear tonal music, we constantly try to find our POSITION in a particular diatonic set with the help of rare interval classes such as 6 and 1. Butler (1989) elaborates this idea and argues that:

> . . . Any tone will suffice as a perceptual anchor—a tonal center—until a better candidate defeats it. The listener makes the perceptual choice of most-plausible tonic on the basis of style-bound conventions in the time ordering of intervals that occur only rarely in the diatonic set; that is, minor seconds (or enharmonics) and the tritone. (238)

In addition, according to Butler and Brown (1981, 1984), we need as few as three pitch classes, two pitches a tritone apart and another single tone as "a reliable aural cue to tonic" (53) to carry out tonic identification judgments.

Although Butler's model is that of key-finding, it seems that POSITION FINDING does not have to be restricted to tonal contexts. The arguments in the rest of this paper presuppose the following hypothesis: If the unique multiplicity property of the diatonic set is the sufficient condition for POSITION FINDING in the diatonic field, it should be also operative in modal contexts, where a FOUND POSITION is related not necessarily to a tonic but to a particular unordered diatonic set.[20] If this is the case, Butler's model has significant implications for atonal pitch organization as well. Dubiel (1991) also suggests that:

> . . . a pitch-class-set analysis of any reasonably complex tonal piece . . . would be bound to involve a distinction between the diatonic collection as presented and the diatonic collection as referred to. The possibility of making such a distinction in non-diatonic contexts should be kept in mind. (1–2)

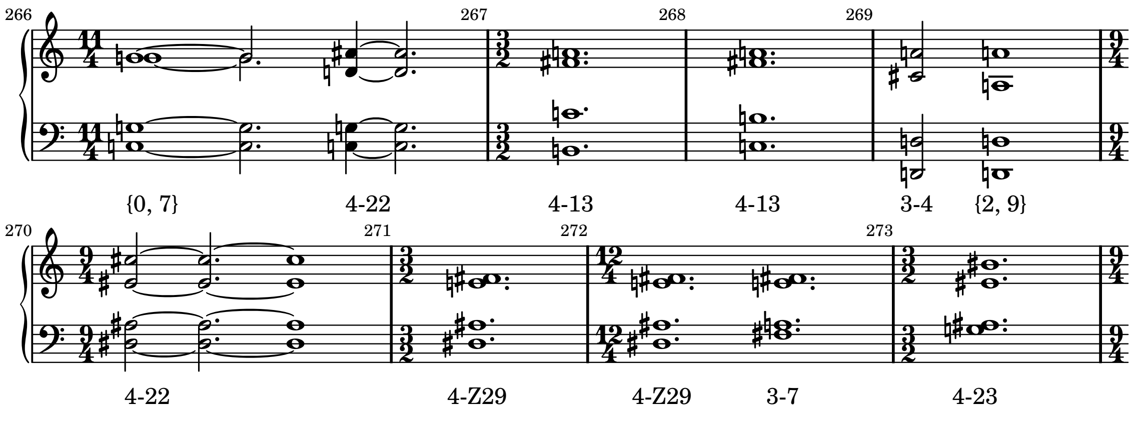
In other words, it seems possible that we still carry out POSITION FINDING in some atonal music in which a diatonic set is not present in its entirety but can be *referred* to with the aid of rare interval classes. Browne's metaphor and Butler's model may well be extended to explain how we perceive some kinds of atonal music.

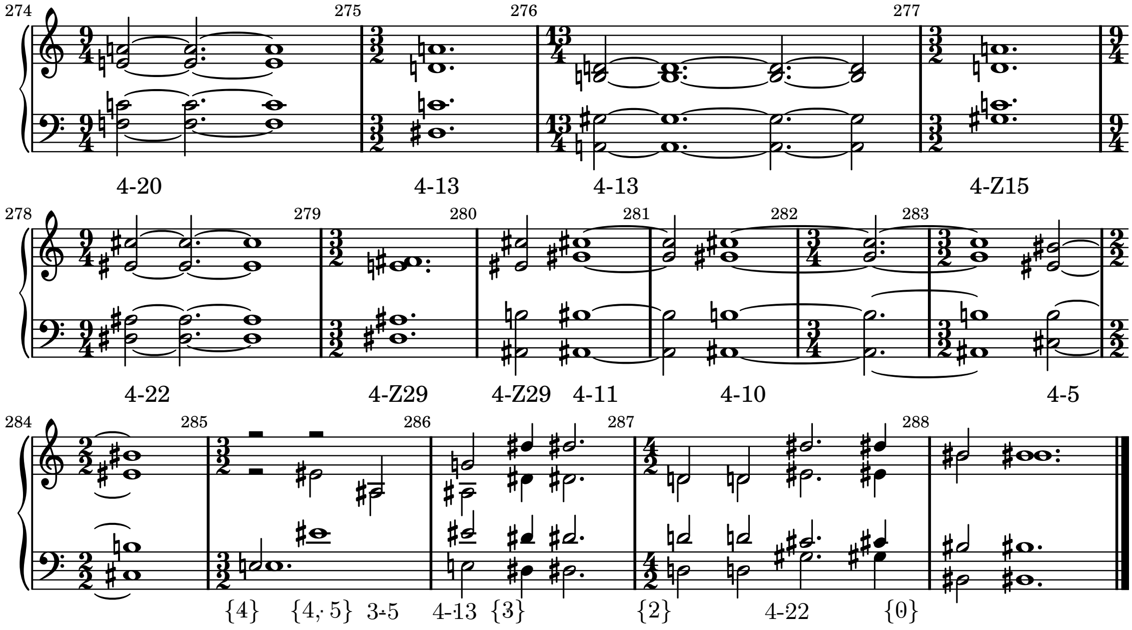
Obviously, there are four different "states" of POSITION FINDING in an atonal passage, that is, a POSITION is FOUND, CONFIRMED, SUGGESTED, or LOST. Example 49 below shows that a POSITION is FOUND with the 4-13s in mm. 275–76 from "Libera me" in *C* scale on $A^\sharp$ and *A* while it is lost with 4-Z15 in m. 277. The example also shows that there are two kinds of "transitions" between POSITIONS, that is, SHIFT, a transposition of a *referred* diatonic set by T_7 or T_5, and LEAP by any other transpositions. Example 52 shows consecutive 3-5 and 4-13 in mm. 285–86 both of which refer to the same diatonic set and hence the second one, 4-13, CONFIRMS the POSITION. Finally, a POSITION is SUGGESTED with a pc-set if it is a proper subset of a diatonic set and the previous state of POSITION is LOST or the previous state is SUGGESTED and a POSITION can not be FOUND even with the union of the two consecutive pc-sets.

Consequently, if the hypothesis is correct, in addition to those proposed by Forte (1973) and Hasty (1981b, 1984), there seems to be another possible guideline for segmentation. That is, on the crudest level, since POSITION FINDING is controlled especially by tritones, segmentation can be carried out so that each pc-set, or what Forte calls "an analytical object," (Forte 1973, 83) in a piece is either a subset of a diatonic collection with a single tritone or else it is not. We may be able to consider these two types of pc-sets as most distinct "analytical objects," which composers, theorists, and listeners treat as building blocks that constitute atonal compositions.

POSITION FINDING AND THE STRUCTURE OF "Libera me"

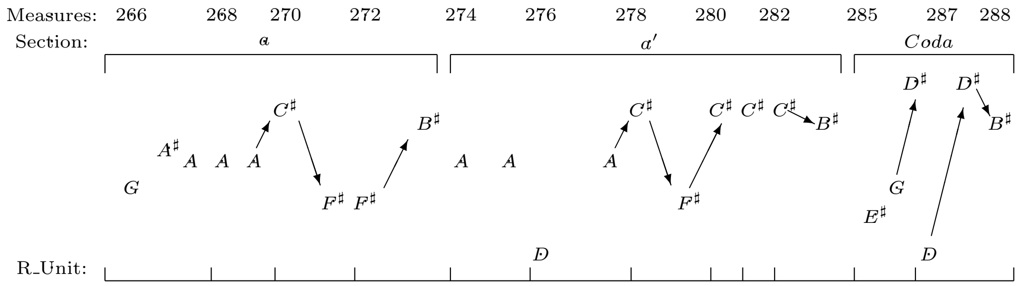
Now, following the hypothetical guideline, let us examine how POSITION FINDING works in Stravinsky's twelve-tone music. The relationships between POSITION FINDING and probabilities of twelve-tone series will be discussed later. Example 43 shows the chords and their note values of the entire movement of "Libera me" from *Requiem Canticles*.





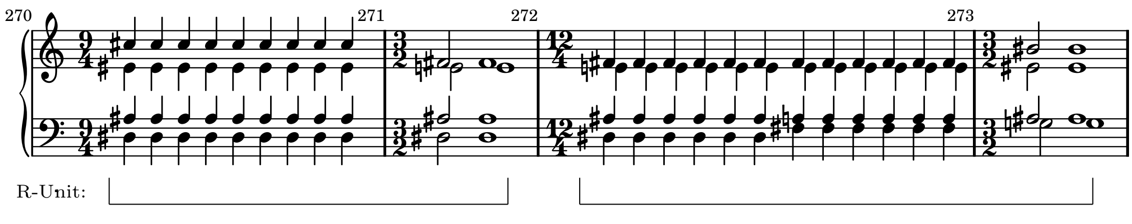
Example 43. The PC-Sets and Their Note Values in "Libera me"

The movement consists of, as Example 44 shows, three sections, namely, *a*, mm. 266–73; a', mm. 274–84; and *Coda*, mm. 285–88 ("R-Unit" stands for Rhythmic Unit). Sections *a* and a' are marked by the repeated soprano contour $A4 - C^\sharp 5 - F^\sharp 4 - (C^\sharp 5) - B^\sharp 4$ and the repeated pair of pc-sets, 4-22 and 4-Z29 in mm. 270–71 and mm. 278-79.



Example 44. Sections and Soprano Contours of "Libera me"

Sections *a* and a' consist of rhythmic units each of which is, as shown in Example 45, a combination of repeated quarter notes and a half and a whole notes.



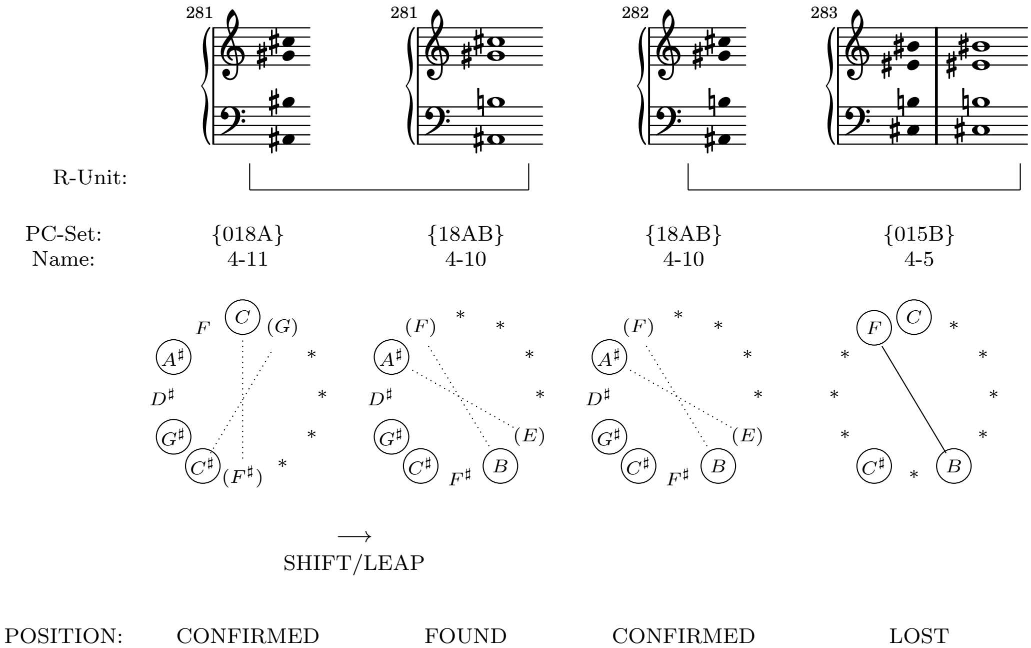
Example 45. Rhythmic Units of "Libera me"

Rhythmic compression occurs in mm. 280–81 as follows:

Example 46. Compressed Rhythmic Units in mm. 280–81

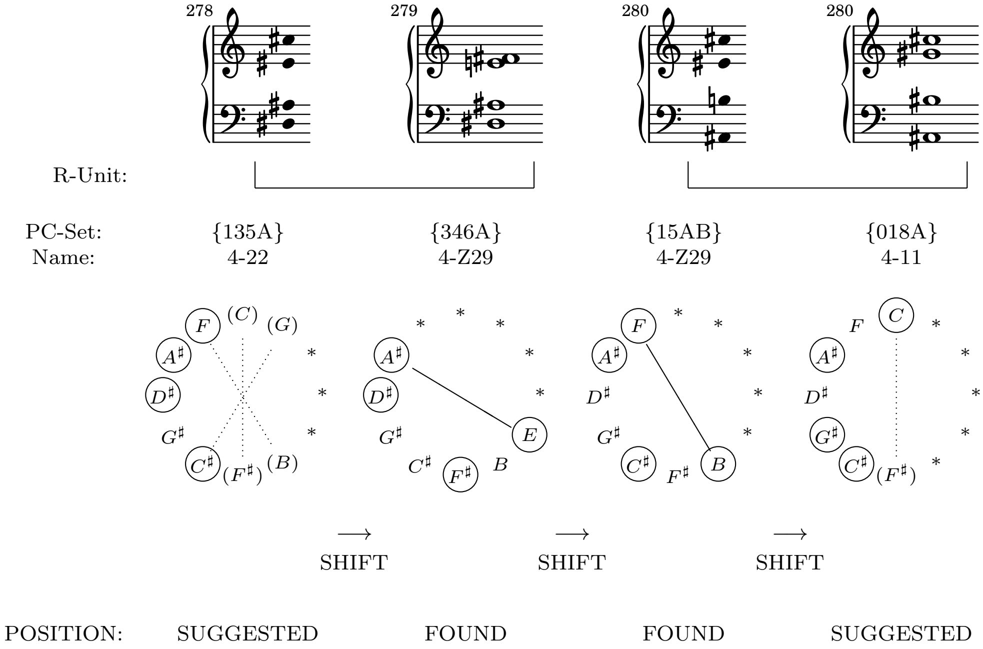
Because the goal of the soprano in each section is $B^\sharp 4$ and this rhythmic compression happens towards the end of section a', the compression adds more tension to the closing of the section than the preceding one and hence prompts the following coda section to be introduced.

In the closing of section a', namely, in mm. 281–84, POSITION FINDING takes place in the following way:



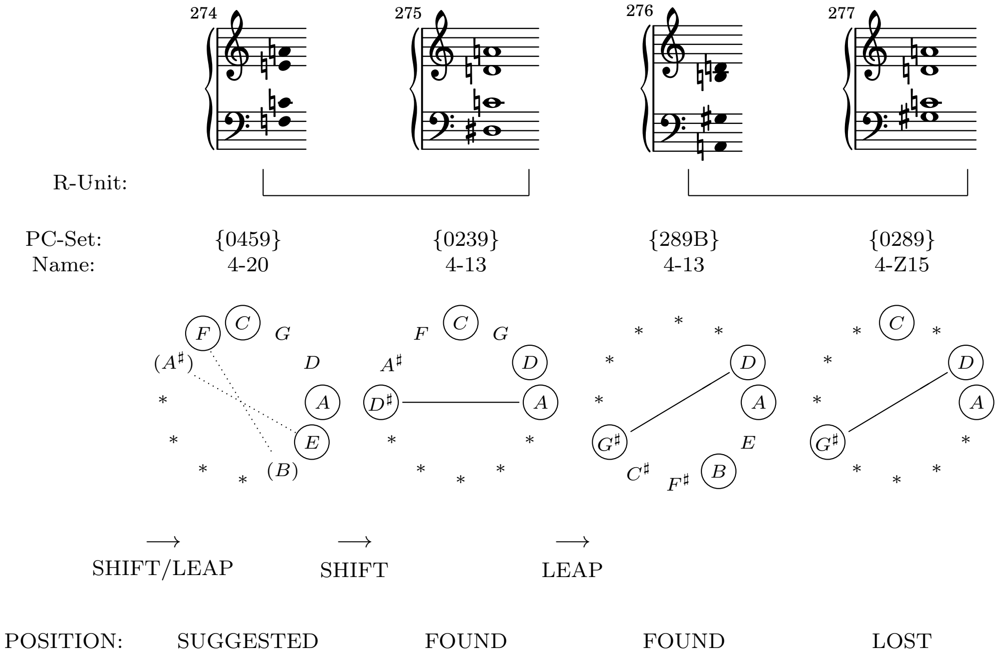
Example 47. POSITION FINDING in mm. 281–84

In the example above, while a line on a circle of 5ths denotes a tritone relation between two pitch classes, a dotted one suggests a possible tritone relation by which a diatonic set is circumscribed. Note that the end of section a' is marked by the LOST POSITION in m. 284. $\{015B\}$ in mm. 283–84, despite its LOST POSITION, might sound like the dominant 7th chord in $F^\sharp$ major with unresolved non-harmonic tone $B^\sharp 4$ in the soprano that should resolve in $C^\sharp 5$ and thus sound like a half cadence. In any case, the uncertainty of the LOST POSITION seems to necessitate the conclusive coda section in mm. 285–88, in which the goal $B^\sharp 4$ is approached twice by the highest pitch $D^\sharp 5$ that are preceded by, as Example 44 shows, ascending motions characteristic of sections a and a'.



Example 48. POSITION FINDING in mm. 278–80

Towards the ending of section a', POSITION FINDING goes as shown in Example 48. Note that contiguous SHIFTS by T_7 in mm. 279–80 also increase tension and move towards the climax at the end of section a'. These SHIFTS seem to cause a DIRECTED MOTION towards the end of the section, which is, as mentioned, enhanced by the rhythmic compression.



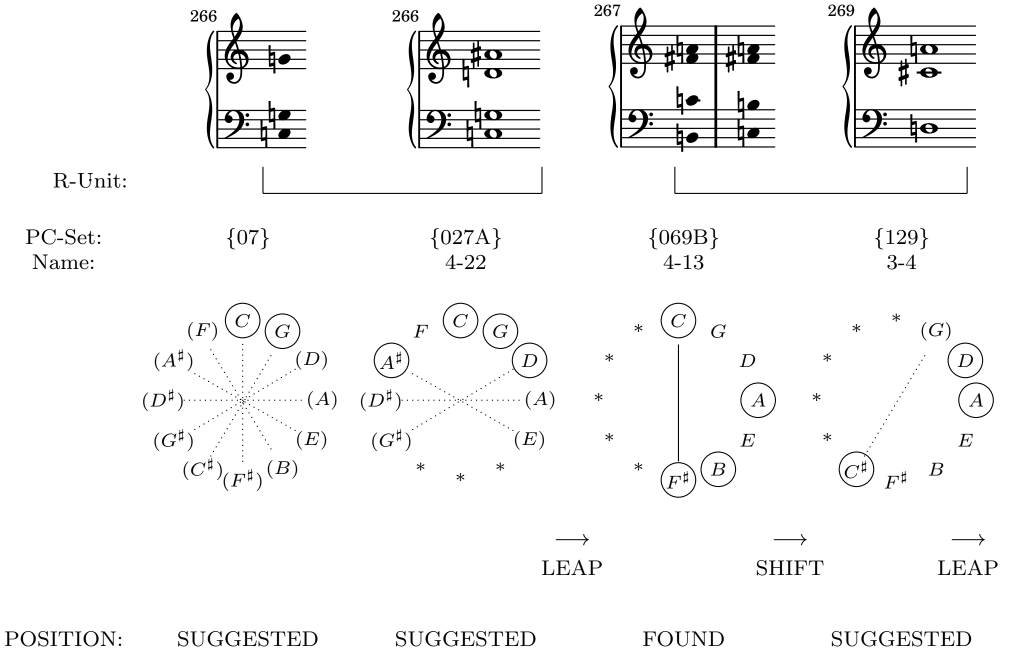
Example 49. POSITION FINDING in mm. 274–77

It may be a case that Stravinsky deliberately set text "... dum discussio venerit, atque ventura ira ..." (when the desolation shall come, and also the coming wrath) to the rhythmic unit in mm. 276–77 shown above that contains the first one of two LOSTs in the entire movement coincides with.

Although, as I argued in the previous section, POSITION FINDING does not have to be related to tonal contexts, {0459} in m. 274 sounds like I^7 (tonic 7th chord) in F major perhaps because the preceding pc-set $\{057A\}$ in m. 273 contains $\{A\}$ so that it forms a tritone with {4} in m. 274 and all the other pitch classes are less than 6 under M_7, by which the C scale on F is referred to, and the chord in m. 274 is a major 7th chord in root position. This “tonic-ness” of the chord sounds appropriate for the beginning of a section, section a'.

Again, the LEAP between m. 275 and m. 276 is accompanied by a melodic leap and initiates the soprano figure. A POSITION is LOST for the first time in m. 277, of which uncertainty seems to necessitate the repetition of the passage in mm. 270–71 in the following measure, m. 278. When a POSITION is LOST, a DIRECTED MOTION seems to be also lost.

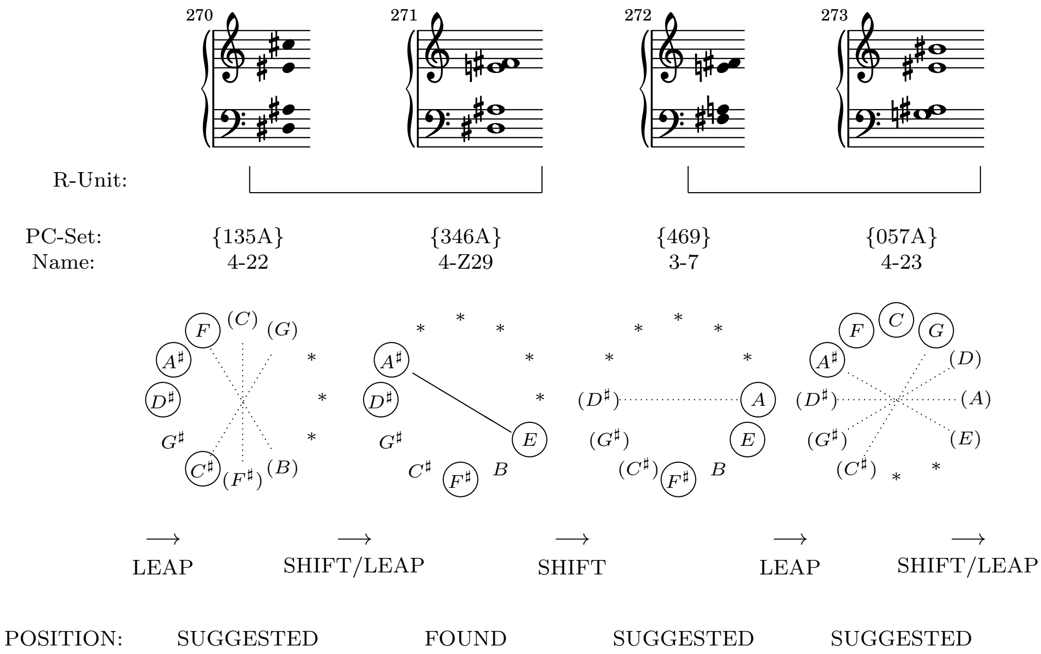
The change from $\{027A\}$ in m. 266 to $\{069B\}$ in m. 267 is a LEAP because the POSITIONS SUGGESTED by $\{027A\}$ are C scale on $D^\sharp$: $\{83A5072\}$, $A^\sharp$: $\{3A50729\}$, or F: $\{A507294\}$, while the POSITION FOUND by $\{069B\}$ is one on G: $\{07294B6\}$.



Example 50. POSITION FINDING in mm. 266–69

$\{129\}$ in m. 269 does not contain a tritone but, since its largest pitch class under M_7 is less than 6, $\{29\}$ is contained in the diatonic set referred to by the preceding pc-set, $\{069B\}$, and only $\{1\}$ is T_7 away from the diatonic set, I assume that tritone $\{17\}$ is latent in m. 269 and the change from $\{069B]$ to $\{129\}$ is a SHIFT.

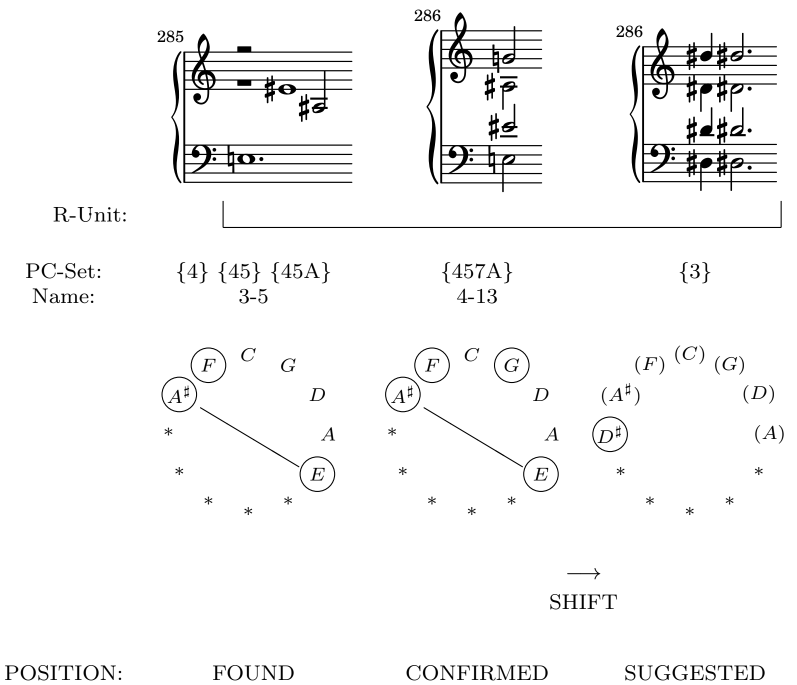
A POSITION LEAP is accompanied by the leap of the soprano line from $A4$ to $C^\sharp 5$ in mm. 269–70 and from $F^\sharp 4$ to $B^\sharp 4$ in mm. 272–73 and the former initiates the characteristic repeated soprano figure mentioned earlier.



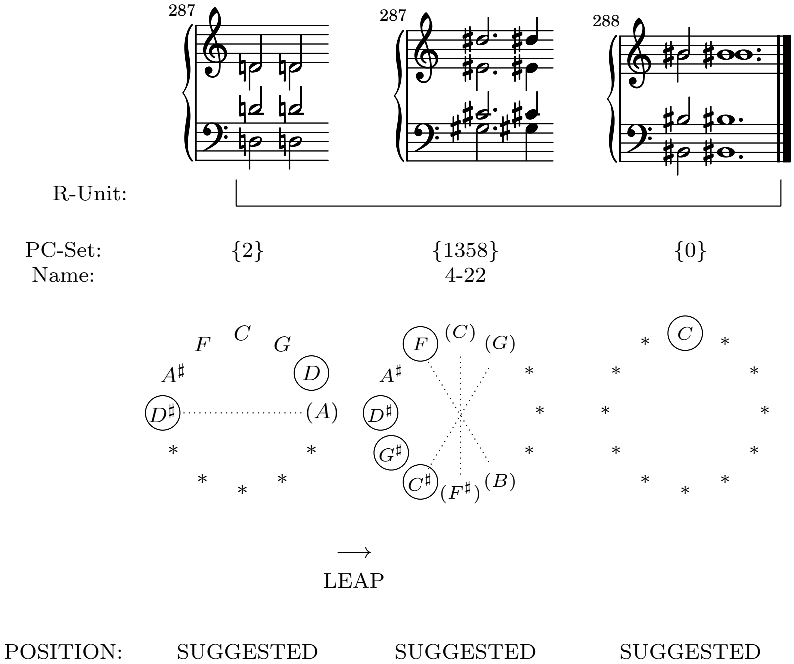
Example 51. POSITION FINDING in mm. 270–73

Contrasting with these LEAPS is the transition from 4-Z29 to 3-7 in mm. 271–72, which exemplifies a readily audible SHIFT.

The beginning of the coda section starting with FOUND and then CONFIRMED POSITIONS leads to a series of SUGGESTED POSITIONS switching from SHIFT to LEAP so that the ending reaches the climax. In the last rhythmic unit of mm. 286–88, the ascending leap from $D4$ to $D^{\sharp}5$ is accompanied, again, by a LEAP so that, although the soprano contour establishes $B^{\sharp}4$ in the last measure as the goal, the entire movement prompts the following concluding movement, "Postlude," to be introduced.



Example 52. The Beginning of Coda in mm. 285–86



Example 53. A Series of SUGGESTED POSITIONS at the End of Coda

To summarize the observations so far: It seems that “Libera me” consists of alternations among different states and transitions of POSITIONS, which correlate with rhythms, melodic contours, and phrase formation and contribute to the formation of the entire movement.

Since many factors correlate with POSITION FINDING, one might think that multi-value logic such as fuzzy logic is useful for describing such relations. I am aware that some theorists try to integrate fuzzy set theory into pc-set theory. I must say, however, that fuzzy logic in music theory remains yet another system of representation. When one employs truth values more than 1, “true,” and 0, “false,” and assigns them to some musical entities, it is necessary to determine exactly where those values are located between 1 and 0, Fuzzy logic or fuzzy set theory itself, however, does not provide *any* guideline to determine their locations. So, again, one needs to rely on some intuitive judgment to decide on the locations before representation with fuzzy logic.

CONCLUSION AND FURTHER ISSUES

The hypothetical guideline of segmentation proposed in the previous section seems to suggest a classification of pc-sets as well. That is, the classification of all the trichords, tetrachords, and pentachords into the following three Tritone-Sensitive Collections, TS-Collections for short, may be useful:

- TS-Collection 1: the subsets of the diatonic set without a tritone
- TS-Collection 2: the subsets of the diatonic set with a tritone
- TS-Collection 3: the rest of the pc-sets

With a pc-set from TS-Collection 1, if a POSITION is already FOUND, what Browne (1981) calls “pattern-matching” happens and the POSITION IS CONFIRMED, if not, POSITIONS ARE SUGGESTED; a pc-set from TS-Collection 2 helps us find our POSITION in a particular diatonic collection; and with one from TS-Collection 3, a POSITION IS LOST.[21]

TS-Collection 1:

Name	T_n/T_nI	M_7	I-vector
4-23	[0257]	[0123]	$< 021030 >$
4-22	[0247]	[0124]	$< 021120 >$
4-26	[0358]	[0134]	$< 012120 >$
4-14	[0237]	[0125]	$< 111120 >$
4-11	[0135]	[0135]	$< 121110 >$
4-20	[0158]	[0145]	$< 101220 >$
4-10	[0235]	[0235]	$< 122010 >$

TS-Collection 2:

Name	T_n/T_nI	M_7	I-vector
4-16	[0157]	[0126]	$< 110121 >$
4-13	[0136]	[0136]	$< 112011 >$
4-Z29	[0137]	[0146]	$< 111111 >$
4-8	[0156]	[0156]	$< 200121 >$
4-27	[0258]	[0236]	$< 012111 >$
4-21	[0246]	[0246]	$< 030201 >$

TS-Collection 3:

Name	T_n/T_nI	M_7	I-vector
4-6	[0127]	[0127]	$< 210021 >$
4-Z15	[0146]	[0137]	$< 111111 >$
4-18	[0147]	[0147]	$< 102111 >$
4-5	[0126]	[0157]	$< 210111 >$
4-9	[0167]	[0167]	$< 200022 >$
4-4	[0125]	[0237]	$< 211110 >$
4-2	[0124]	[0247]	$< 221100 >$
4-1	[0123]	[0257]	$< 321000 >$
4-17	[0347]	[0347]	$< 102210 >$
4-19	[0148]	[0148]	$< 101310 >$
4-7	[0145]	[0158]	$< 201210 >$
4-24	[0248]	[0248]	$< 020301 >$
4-12	[0236]	[0258]	$< 112101 >$
4-25	[0268]	[0268]	$< 020202 >$
4-3	[0134]	[0358]	$< 212100 >$
4-28	[0369]	[0369]	$< 004002 >$

Example 54. TS-Collections of T_n/T_nI Tetrachords

Now, consider again the desirability scale of the tetrachords shown in Example 42. 4-Z29 and 4-13 come from Collection 2 and 4-22 from Collection 1, which have higher desirabilities while 4-2, 4-6, and 4-18, which have negative desirabilities, come from Collection 3. It seems, therefore, plausible to assume one of properties 1 and 2 that the largest member of the T_n/T_nI type under M_7 is 6 or less. This may be the reason why 4-Z15 has a lower desirability than its Z-related counterpart 4-Z29. The former has pitch-class 7 under M_7 as its largest member.

In tonal music, we always try to find our POSITIONS with respect to the tonic and the diatonic collection in the main key, which is the absolute referential point. Also, as Browne (1981) points out, because of the unique multiplicity property, "the various transpositions are hierarchically related to the referential set [in the tonic key] by their various common-tone distributions" (Browne 1981) and the distance from the set can be measured with relative ease. By contrast, the observations so far seem to suggest that, in *some* kinds of twelve-tone and atonal music such as "Libera me," implied diatonic collections are, of course, not related to a single referential diatonic collection. Instead, it seems that those implied diatonic collections adjacent in a time dimension are related to each other in terms of their relative distances on the circle of fifths and, in atonal music of those kinds, pitches are organized in such a way that we can FIND POSITIONS. Thus I would like to regard the chromaticism in "Libera me" as quick transpositions of referred diatonic collections.

Segmentation of pc-sets as simultaneities may be in part justified because of the mere fact that simultaneously sounding pc-sets can not be dissolved in the time dimension and thus they are basic elements for POSITION FINDING. However, as Butler and Brown (1981) and Brown, Butler, and Jones (1994) point out, temporal orders of pitch classes affect POSITION FINDING. So, segmentation in terms of simultaneity is a rather crude way of identifying pc-sets. Needless to say, some pitches in a piece are more salient than others due to their registers, dispositions in a chord, dynamics, durations, and so on. In addition, Butler and Brown (1981) and Brown, Butler, and Jones (1994) point out that temporal orders of pitch classes affect POSITION FINDING. In this respect, Narmour's Implication-Realization Model (1990, 1992) may be useful to incorporate POSITION FINDING with other factors. Thus, we need to further calibrate the degrees of implications for POSITIONS by adding more criteria.

APPENDIX

If Stravinsky had some particular properties in mind and looked for tetrachords which possessed them, the relative frequencies of the tetrachordal subsets are all 1/21. In this case, the probabilities of all the 21 T_n/T_nI tetrachords are as follows:

Since

$$P(\text{4-1}) = {}_{17}C_0 \left(\frac{1}{21}\right)^0 \left(\frac{20}{21}\right)^{17} = 0.436297$$
$$P(\text{4-5}) = {}_{17}C_1 \left(\frac{1}{21}\right)^1 \left(\frac{20}{21}\right)^{16} = 0.370852$$
$$P(\text{4-Z29}) = {}_{17}C_3 \left(\frac{1}{21}\right)^3 \left(\frac{20}{21}\right)^{14} = 0.037085$$
$$P(\text{4-13}) = {}_{17}C_4 \left(\frac{1}{21}\right)^4 \left(\frac{20}{21}\right)^{13} = 0.006490$$

therefore,

Name	R_Frq	Frq	$B(n,p)$				
4-1	1/21	0	0.436297	4-14	1/21	0	0.436297
4-2	1/21	0	0.436297	4-Z15	1/21	1	0.370852
4-3	1/21	0	0.436297	4-16	1/21	0	0.436297
4-4	1/21	0	0.436297	4-18	1/21	0	0.436297
4-5	1/21	1	0.370852	4-20	1/21	1	0.370852
4-6	1/21	0	0.436297	4-22	1/21	4	0.006490
4-7	1/21	0	0.436297	4-23	1/21	1	0.370852
4-10	1/21	1	0.370852	4-26	1/21	0	0.436297
4-11	1/21	1	0.370852	4-27	1/21	0	0.436297
4-12	1/21	0	0.436297	4-Z29	1/21	3	0.037085
4-13	1/21	4	0.006490				
					Total:	17	

	Satisfy principles	Not satisfy principles
4-13 4 times	0.006490	0.993510
4-Z29 3 times	0.037085	0.962915
4-5 once	0.370852	0.629148
¬4-1	0.436297	0.563703

$$p + 0.006490e = q + 0.037085e = r + 0.370852e = 0.436297e$$

$$p = 0.429807e$$
$$q = 0.399212e$$
$$r = 0.065445e$$

	Satisfy principles	Not satisfy principles
4-13 4 times	1.429807	0.429807
4-Z29 3 times	1.399212	0.399212
4-5 once	1.065445	0.065445
¬4-1	1.000000	0.000000

Accordingly, the desirability scale will be the following:

Choice	Desirability	Consequence
Properties 1	1.429807	4-13 or 4-22 4 times
Properties 2	1.399212	4-Z29 3 times
Properties 3	1.065445	4-5, 4-10, 4-11, 4-Z15, 4-20, or 4-23 once
Properties 4	1.000000	¬4-1, ¬4-2, ¬4-3, ¬4-4, ¬4-6, ¬4-7, ¬4-12, ¬4-14, ¬4-16, ¬4-18, ¬4-26, nor ¬4-27

NOTES

[1]Earlier versions of this paper were presented as "Stravinsky's 12-Tone Music in a Bayesian Framework" at the Annual Meeting of the Japanese Society for Aesthetics, Kyoto City College of the Arts, Kyoto, Japan, October, 2000, and "Diatonic Implications of Atonal Pitch Organization" at the Society for Music Perception and Cognition Conference, Queen's University, Kingston, Canada, August, 2001.

[2]Issues of segmentation have been discussed by Cambouropoulos (2006), Forte (1972, 1973, 1988), Hasty (1981a, 1981b, 1984, 1986), Hanninen (2001, 2004), Lartillot (2004), and Tenney and Polansky (1980).

[3]Different notions of "probability" will be discussed later.

[4]For an application of Bayesian principles to the analysis of all the twelve-tone compositions by Stravinsky, refer to Takaoka (1999).

[5]Twelve-tone techniques along with properties of twelve-tone series have been discussed by Babbitt (1955, 1960), Morris (1977, 1983–84), Morris and Starr (1974), and so on. A number of researchers have discussed pitch organization in atonal music and proposed various theories; Salzer (1962) and Travis (1959, 1966, 1970), for example, applied Schenkerian theory, although this approach invited a compelling criticism from Straus (1987); Newlin (1974) and Ogdon (1981) conducted Roman-numeral analyses. Perle (1955, 1977, 1991, 1992) discusses symmetric pitch relations in some atonal music, Lewin (1968a) proposes the principle of "inversional balance," and Cohn (1988, 1991b) emphasizes the special role transpositionally related pitch-class sets play.

[6]Issues of the relations between choices of a twelve-tone row and pitch materials derived from it are discussed by Hasty (1988).

[7] ${}_6C_4$ stands for the number of possible combinations of 4 elements out of 6. In general, ${}_nC_r = n!/r!(n-r)!$.

[8]The derivation of these pc-sets from the rotational array will be discussed later.

[9] $x^2 = \sum_{i=1}^{15}(n_i - Np_i)^2/Np_i = 26.714$

[10]While the pitch organization in Stravinsky's post-tonal, pre-serial compositions has been discussed by a considerable number of music scholars such as Antokoletz (1986), Berger (1968), Cone (1963), Forte (1973, 1978a), Straus (1982a, 1982b), Taruskin (1990, 1996), Travis (1959), and Van den Toorn (1983, 1987) to name a few, his twelve-tone music has been paid attention only in Babbitt (1986), Clemmons (1977), Clifton (1970), Keller (1955), Hasty (1984), Kohl (1979 and 1980), Phillips (1984), Spies (1965a, 1965b, 1967), Straus (1997, 1999a, 1999b, 2001), Van den Toorn (1983, 1987), Ward-Steinman (1961), Watkins (1986), White (1979), Wuorinen and Kresky (1986), and Rust (1994). Among Stravinsky's 15 twelve-tone compositions, only seven pieces, namely, *Canticum Sacrum*, *Agon*, *The Flood*, *Abraham and Isaac*, *Variations*, *Introitus*, and *Requiem Canticles*, have been analyzed in depth, by Kohl (1979 and 1980), Phillips (1984), Spies (1965a, 1965b, 1967), and van den Toorn (1983). In short, Stravinsky's twelve-tone practice in his entire repertoire of twelve-tone music has not yet been examined.

[11]Since "dependencies" in the case of twelve-tone music are related to but not limited to inclusion relationships between twelve-tone rows or hexachords and derived pc-sets, the notion of set complexes, proposed by Forte (1964, 1973), is useful for studies in dependencies among compositional choices.

[12]As Straus (2001, 152) points out, these hexachords actually come from the rows of the rotational arrays of P1.

[13]Joseph Straus argues in personal communication that Stravinsky always treats the hexachords as ordered lines and derives rotational arrays from them; Each of the rows of the array is also an ordered line and Stravinsky very rarely deviates from that ordering; Beginning with Movements, with a few very local exceptions, Stravinsky never transposes a series; All of the material is derived from the hexachords of four basic forms (P, I, R, IR); The only transposition takes place within the rotational arrays. It seems to me, however, those deviations and transpositions are not "rare" but "many" and very significant. Since Stravinsky deviated and transposed anyway, I assume that hexachords and twelve-tone series forms in their all 48 forms were available to him. Needless to say, the probabilistic model based on Prof. Straus' standpoint will provide us with another important perspective.

[14]Spies (1967, 112) also points out that "...the texts [of *Requiem Canticles*] were to be segments of, or sentences from — rather than liturgically complete — prayers.

[15]There are several different types of decision theory. For example, Jeffrey (1983, 59) points out the difference between Ramsey's theory and his as follows: "Ramsey attributes desirabilities to consequences but attributes probabilities to . . . possible conditions . . . . . . developed an alternative to Ramsey's system: a theory of preference which is unified in the sense that it attributes probabilities and desirabilities to the same objects, and noncausal . . . ."

[16]The following discussion is largely based on Oide 1980. For a historical account of the notion of probability, see Hacking (1975, 57–101)

[17]The probabilistic model of the choice of the tetrachords discussed here might not be plausible because, after choosing a certain tetrachord some times, Stravinsky might avoid choosing it again. This issue has to do with that of conditional probability, which is discussed by Takaoka (1999).

[18]If Stravinsky had some particular properties in mind and looked for tetrachords which possessed them, the relative frequencies of the tetrachordal subsets are all 1/21. The desirability scale in this case is presented in APPENDIX.

[19]Following the convention adopted by Lakoff and Johnson (1980), metaphorical expressions are henceforth written in capital letters.

[20]For the distinction between key- and diatonic-set-finding, see Butler 1998 and Brown 1988.

[21]Van Egmond and Butler (1997) classify T_n types with respect to the degrees of implications of tonal centers as well as major and minor modes.

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